Algebraic Geometry Without Tears

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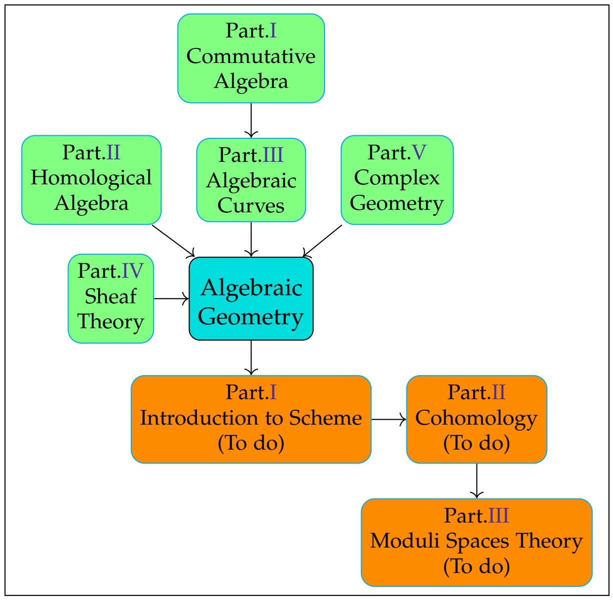
The Structure of AG



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Chapter 1

Some

1.1 1.2

Definition
A class is called **small** if it has a cardinal number.

Theorem
A class is a set iff it is small.

Definition
A class that is not a set is called a **proper class**.

Example.
$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are sets, the collection of all sets is a proper class, and the collection R of all Russell classes is not even a class.

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Part I

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Chapter 1

Beginning Concepts

1.1 The Definition of Algebraic Varieties

1.1.1 Affine Algebraic Sets

1.1.2 Zariski Topology

Algebraic geometry builds fundamental concepts of geometry out of pure algebra (rings and polynomials). A very basic concept of geometry is topology.

This is definition of topological space.

Chapter 2

Introduction to Scheme

2.1 The Definition of Algebraic Varieties

2.1.1 Affine Algebraic Sets

Systems of polynomial equations in variables $x_1, ..., x_n$ can always be written in the form

$$p_1(x_1, ..., x_n) = 0$$

$$...$$

$$p_m(x_1, ..., x_n) = 0$$

where $p_1, ..., p_m$ are polynomials.

Solutions of the above equations are n-tuples of elements $(x_1, ..., x_n)$ of the given field which satisfy the equations.

Such n-tuples are also called zeros of the polynomials $p_1, ..., p_n$.

Sets of zeros of sets of polynomials are called affine algebraic sets.

Solutions of the above equations, or zeros of the polynomials $p_1, ..., p_m$, are also zeros of all linear combinations $a_1 p_1 + ... + a_m p_m$ where $a_1, ..., a_m$ are arbitrary polynomials.

The elements form the ideal generated by $p_1, ..., p_m$, which is denoted by $(p_1, ..., p_m)$.

An ideal in a commutative ring is a subset which contains 0, is closed under $+$, and multiples by elements of the ring.

The ring of polynomials is n variables over a field is Noetherian, which means that every ideal is finitely generated (i.e. generated by finitely many elements).

Note: A commutative ring R is Noetherian iff it satisfy the ascending chain condition (ACC) with respect to ideals.

2.1.2 Complex Numbers

Every non-constant polynomial in one variables with coefficients in $\mathbb{C}$ has at least one zero (we also say root). A field which satisfied this property is called algebraically closed.

The fact that $\mathbb{C}$ is algebraically closed is known as the fundamental theorem of algebra.

Definition
Let A be a field. Then an affine n-space $A^n = \{x = (x_1, x_2, \ldots, x_2) \mid x_i \in A, \forall i = 1, 2, \ldots, n\}$ is a vector space of dimension n over the field A.

Example.
The Euclidean n-space $\mathbb{R}^n$ is an affine n-space of dimension n.

Definition
Given an affine space A^n, an n-tuple $x = (x_1, x_2, \ldots, x_n) \in A^n$ is said to be a zero of a polynomial $f(x) = f(x_1, x_2, \ldots, x_n) \in A[x_1, x_2, \ldots, x_n]$ (polynomial ring of n determinates over A), if $f(x) = f(x_1, x_2, \ldots, x_n) = 0$.
Given a subset $S \subseteq A[x_1, x2, \ldots, x_n]$, the algebraic set $V(S)$ of zeros of S is defined by $V(S) = \{x \in A^n \mid f(x) = 0, \forall f \in S\} \subseteq A^n$.
A subset $X \subseteq A^n$ is said to be an affine algebraic set or simply, an **algebraic set** if there is a subset $S \subseteq A[x_1, x_2, \ldots, x_n]$ such that $X = V(S)$.

Definition
Let X be a subset of A^n and the subset $I(X)$ of $A[x_1, x_2, \ldots, x_n]$ be defined by $I(X) = \{f \in A[x_1, x_2, \ldots, x_n] \mid f(x) = 0 \text{ for all } x \in A\}$, is an ideal of $A[x_1, x_2, \ldots, x_n]$, called the ideal of X.

Definition
An algebraic set X in A^n is said to be an **affine variety** if $I(X)$ is a prime ideal in the polynomial ring $A[x_1, x_2, \ldots, x_n]$.

2.2 Locally Ringed Spaces

Definition (Scheme)
A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ which admits an open covering $X = \bigcup_{i \in I} U_i$ s.t. all locally ringed spaces $(U_i, \mathcal{O}_X|_{U_i})$ are affine schemes.

Definition (Scheme of second definition)
A **scheme** is a locally ringed space with the property that every point has an open neighborhood which is an affine scheme.

A morphism of schemes is a morphism of locally ringed spaces. The category of schemes will be denoted Sch.

Definition (Affine Scheme)
Locally ringed spaces isomorphic to $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ will be called affine schemes.

Definition (Affine Scheme of second definition)
An **affine scheme** is a locally ringed space isomorphic as a locally ringed space to $\mathrm{Spec}(R)$ for some ring R.

A morphism of affine schemes is a morphism in the category of locally ringed spaces.

Definition (The spectrum of a ring)
Let R be a ring. The spectrum of R is the set of prime ideals of R. It is usually denoted $\mathrm{Spec}(R)$.

Definition (Ringed Space)
A ringed space is a topological space X with a sheaf of rings $\mathcal{O}_X$. The sheaf $\mathcal{O}_X$ is called the structure sheaf of the ringed space $(X, \mathcal{O}_X)$.

A ringed space $(X, \mathcal{O}_X)$ is called a local ringed space if $\mathcal{O}_X$ is a sheaf of local rings.

Definition (Ringed Space)
A ringed space is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of unital rings.

If all stalks of the structure sheaf are local rings, it is called a locally ringed space.

Definition (Locally Ringed Space)
A locally ringed space $(X, \mathcal{O}_X)$ is a pair consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ all of whose stalks are local rings.
A locally ringed space is a ringed space $(X, \mathcal{O}_X)$ s.t. the stalks of the structure sheaf $\mathcal{O}_X$ are local rings.

2.3 Moduli spaces

This section discusses morphisms $f \colon \mathcal{X} \to Y$ from algebraic stacks to algebraic spaces. Under suitable hypotheses Y is called a *moduli space* for $\mathcal{X}$. If $\mathcal{X} = [U/R]$ is a presentation, then we obtain an R-invariant morphism $U \to Y$ and (under suitable hypotheses) Y is a *quotient* of the groupoid (U, R, s, t, c). A discussion of the different types of quotients can be found starting with Quotients of Groupoids.

Definition
Let $\mathcal{X}$ be an algebraic stack. Let $f \colon \mathcal{X} \to Y$ be a morphism to an algebraic space Y.

1. We say f is a *categorical moduli space* if any morphism $\mathcal{X} \to W$ to an algebraic space W factors uniquely through f.
2. We say f is a *uniform categorical moduli space* if for any flat morphism $Y' \to Y$ of algebraic spaces the base change $f' \colon Y' \times_Y \mathcal{X} \to Y'$ is a categorical moduli space.

Let $\mathcal{C}$ be a full subcategory of the category of algebraic spaces.

3. We say f is a *categorical moduli space in* $\mathcal{C}$ if $Y \in \mathrm{Obj}(\mathcal{C})$ and any morphism $\mathcal{X} \to W$ with $W \in \mathrm{Obj}(\mathcal{C})$ factors uniquely through f.
4. We say is a *uniform categorical moduli space in* $\mathcal{C}$ if $Y \in \mathrm{Obj}(\mathcal{C})$ and for every flat morphism $Y' \to Y$ in $\mathcal{C}$ the base change $f' \colon Y' \times_Y \mathcal{X} \to Y'$ is a categorical moduli space in $\mathcal{C}$.

By the Yoneda lemma a categorical moduli space, if it exists, is unique. Let us match this with the language introduced for quotients.

Lemma
Let (U, R, s, t, c) be a groupoid in algebraic spaces with $s, t\colon R \to U$ flat and locally of finite presentation. Consider the algebraic stack $\mathcal{X} = [U/R]$. Given an algebraic space Y there is a 1-to-1 correspondence between morphisms $f\colon \mathcal{X} \to Y$ and R-invariant morphisms $\phi\colon U \to Y$.

Lemma
With assumption and notation as in Lemma. Then f is a (uniform) categorical moduli space iff ϕ is a (uniform) categorical quotient. Similarly for moduli spaces in a full subcategory.

Lemma
Let $f\colon \mathcal{X} \to Y$ be a morphism from an algebraic stack to an algebraic space. If for every affine scheme Y' and flat morphism $Y' \to Y$ the base change $f'\colon Y' \times_Y \mathcal{X} \to Y'$ is a categorical moduli space, then f is a uniform categorical moduli space.

Part II

Cohomology

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Part III

Moduli Spaces Theory

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